Journal of Engineering Mathematics, Vol. II, No. 2, April 1968.
Wolters-Noordhoff Publishing - Groningen.
Printed in the Netherlands.

PRIMARY CREEP IN THICKWALLED SHELLS

by

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SUMMARY

An analytic study of the problem of primary creep in thickwalled spherical and cylindrical shells is given. Particular emphasis is placed on the asymptotic phase and an explicit expression for the additional displacement, due to statical primary creep, is obtained. A numerical method for both primary and secondary creep is outlined.

1. *Introduction.*

The problem of primary creep in spherical and cylindrical shells, with an internal pressure applied instantaneously, has been treated by several authors. A numerical method for the cylindrical case has been given by Besseling [1], but in the numerical results presented, the material exhibits only secondary creep. Analytic studies have been given by Hult [2] and Rosengren [3]. In [2] the initial phase is considered, and an expansion valid for small times is given. The asymptotic case for a thinwalled spherical shell with only secondary creep is treated with a perturbation method. This method was extended by Rosengren to include also primary creep in cylindrical vessels.

2. *Basic equations for spherical shells.*

The total strain tensor ϵ_{ij} is the sum of the elastic strain tensor $\epsilon_{ij}^{(e)}$ and the creep strain tensor $\epsilon_{ij}^{(c)}$. The elastic strain is governed by Hooke's law

$$\epsilon_{ij}^{(e)} = \frac{1+\mu}{E}\,\sigma_{ij} - \frac{\mu}{E}\,\sigma_{kk}\,\delta_{ij}\,, \tag{1}$$

while the creep strain is assumed to satisfy

$$\frac{\partial\epsilon_{ij}^{(c)}}{\partial t} = \frac{3}{2}\,K\,\frac{\sigma_e^{n-1}}{\epsilon_e^{(c)m}}\,\cdot\,s_{ij}\,, \tag{2}$$

cf. Odqvist and Hult [4]. The quantities K, m and n are non-negative constants with $\lambda = n/(m+1) \geqslant 1$ for a large number of metals. The case $m = 0$ applies to secondary creep. The stress deviation tensor s_{ij}, the effective stress σ_e and the effective creep strain $\epsilon_e^{(c)}$ are defined by

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\,\sigma_{kk}\,\delta_{ij} \tag{3}$$

$$\sigma_e^2 = \frac{3}{2} s_{ij} \cdot s_{ij} \tag{4}$$

$$\epsilon_e^{(c)2} = \frac{2}{3} \epsilon_{ij}^{(c)} \cdot \epsilon_{ij}^{(c)} \tag{5}$$

The spherical shell, which is loaded by an internal pressure p, is characterized by spherically symmetric stress and strain tensors, depending only on radius r and time t. The principal directions are indexed r, t, t (radial and tangential). The strain components are

$$\epsilon_r = \frac{\partial u}{\partial r} \tag{6}$$

and

$$\epsilon_t = \frac{u}{r}, \tag{7}$$

where u(r, t) is the radial displacement of a particle initially at radius r. The equilibrium conditions give

$$\frac{\partial \sigma_r}{\partial r} - \frac{2}{r} (\sigma_t - \sigma_r) = 0. \tag{8}$$

The initial condition is

$$\epsilon_{ij}^{(c)}(r, 0) = 0, \tag{9}$$

while the obvious boundary conditions are

$$\sigma_r (a, t) = - p \tag{10}$$

$$\sigma_r (b, t) = 0, \tag{11}$$

where a and b are inner and outer radius of the sphere.
The problem defined by equations (1) - (11) will now be solved.

Since there is no creep strain at the first moment, the Lamé solution is valid as initial conditions for σ_{ij} and ϵ_{ij},

$$\sigma_r (r, 0) = - p \frac{r^{-3} - b^{-3}}{a^{-3} - b^{-3}} \tag{12}$$

$$\sigma_t (r, 0) = p \frac{\frac{1}{2} r^{-3} + b^{-3}}{a^{-3} - b^{-3}} \tag{13}$$

$$u(a, 0) = \frac{pa}{2E} \cdot \frac{(1+\mu) a^{-3} + 2(1-2\mu) b^{-3}}{a^{-3} - b^{-3}} . \tag{14}$$

From (2) follows

$$\frac{\partial \epsilon_{kk}^{(c)}}{\partial t} = 0$$

and hence from (9)

$$\epsilon_{kk}^{(c)} = 0, \tag{15}$$

expressing the creep strain incompressibility. We now consider the total strain sum

$$\epsilon_{kk} = \epsilon_{kk}^{(c)} + \epsilon_{kk}^{(e)} = \epsilon_{kk}^{(e)} = \frac{1-2\mu}{E}\sigma_{kk} \tag{16}$$

where we have used (1). By equations (6) and (7) we obtain

$$\frac{\partial u}{\partial r} + 2\frac{u}{r} = \frac{1-2\mu}{E}\sigma_{kk}. \tag{17}$$

We now use the equilibrium equation (8) to express the sum σ_{kk} in terms of $\sigma_r(r,t)$, which gives

$$\frac{\partial u}{\partial r} + 2\frac{u}{r} = \frac{1-2\mu}{E}\left[3\sigma_r + r\frac{\partial\sigma_r}{\partial r}\right]. \tag{18}$$

The radial displacement is thus given by

$$u(r,t) = \frac{1-2\mu}{E} r\,\sigma_r(r,t) + \frac{F(t)}{r^2}, \tag{19}$$

with F(t) as unknown integration constant. From the stress tensor we obtain the effective stress

$$\sigma_e = \sigma_t - \sigma_r, \tag{20}$$

and with (8) we get

$$\sigma_e = \frac{1}{2} r\frac{\partial\sigma_r}{\partial r}. \tag{21}$$

We now express the remaining quantities of (2) in terms of σ_r and F(t). The tangential creep strain $\epsilon_t^{(c)}$ is given by

$$\epsilon_t^{(c)} = \epsilon_t - \epsilon_t^{(e)} = \frac{u}{r} - \frac{1+\mu}{E}\sigma_t + \frac{\mu}{E}\sigma_{kk}. \tag{22}$$

Equations (8) and (19) give

$$\epsilon_t^{(c)} = \frac{F(t)}{r^3} + \frac{1}{2E}(\mu-1)\, r\frac{\partial\sigma_r}{\partial r}. \tag{23}$$

We now consider the stress deviation s_t and get

$$s_t = \sigma_t - \frac{1}{3}\sigma_{kk} = \frac{1}{3}(\sigma_t - \sigma_r) \quad \frac{r}{6}\frac{\partial\sigma_r}{\partial r}. \tag{24}$$

The only remaining quantity of (2) is $\epsilon_e^{(c)}$, which is obtained from

$$\epsilon_e^{(c)2} = \frac{2}{3}\left(\epsilon_r^{(c)2} + 2\,\epsilon_t^{(c)2}\right). \tag{25}$$

From (15) follows

$$e_{kk}^{(c)} = \epsilon_r^{(c)} + 2\,\epsilon_t^{(c)} = 0, \tag{26}$$

and hence (25) takes the form

$$\epsilon_e^{(c)} = 2\,\epsilon_t^{(c)}. \tag{27}$$

We now consider (2) in the tangential direction and obtain

$$\frac{\partial}{\partial t}\left[\frac{F(t)}{r^3} + \frac{1}{2E}\,(\mu-1)\,r\frac{\partial\sigma_r}{\partial r}\right]^{m+1} = (m+1)\cdot K\cdot 2^{-(n+m+1)}\cdot\left(r\,\frac{\partial\sigma_r}{\partial r}\right)^n. \tag{28}$$

This differential equation together with the boundary conditions (10) and (11) and the initial conditions (12) and (14) defines a mixed initial-boundary value problem for the functions $\sigma_r\,(r,t)$ and $F(t)$.

3. *Equations for cylindrical shells.*

The derivation of the equation corresponding to (28) in the case of internal pressure in a cylindrical vessel has been given by Rosengren [3] in the case of Poisson's ratio $\mu = 1/2$. With earlier symbols we get

$$\frac{\partial}{\partial t}\left[\frac{f(t)}{r^2} - \frac{3}{4E}\,r\,\frac{\partial\sigma_r}{\partial r}\right]^{m+1} = (m+1)\cdot K\cdot\left(\frac{3}{4}\right)^{\frac{n+m+1}{2}}\cdot\left(r\,\frac{\partial\sigma_r}{\partial r}\right)^n, \tag{29}$$

where

$$f(0) = \frac{3}{2}\cdot\frac{p}{E}\cdot\frac{1}{a^{-2} - b^{-2}} \tag{30}$$

$$\sigma_r(r,0) = p\,\frac{b^{-2} - r^{-2}}{a^{-2} - b^{-2}} \tag{31}$$

$$\sigma_r(a,t) = -\,p \tag{32}$$

$$\sigma_r(b,t) = 0. \tag{33}$$

The other principal stresses are given by

$$\sigma_t(r,t) = \sigma_r(r,t) + r\,\frac{\partial\sigma_r}{\sigma r} \tag{34}$$

and

$$\sigma_z(r,t) = \sigma_r(r,t) + \frac{1}{2} r \frac{\partial \sigma_r}{\partial r} . \tag{35}$$

The radial displacement is defined by

$$u(r,t) = \frac{1}{r} . f(t) . \tag{36}$$

We now introduce some new definitions in order to avoid unnecessary constants.

$$\sigma(x,\tau) = \frac{1}{p} . \sigma_r(r,t) \tag{37}$$

$$g(\tau) = \frac{4}{3} \frac{E}{p} \frac{1}{a^2} f(t) \tag{38}$$

$$\theta = (m+1) \left(\frac{3}{4}\right)^{\frac{n-m-1}{2}} .K.p^n . \left(\frac{E}{p}\right)^{m+1} . t \tag{39}$$

$$\tau = \theta^{\frac{1}{m+1}} . \left(\frac{2}{\lambda} \frac{c^{2/\lambda}}{c^{2/\lambda}-1}\right)^{\lambda-1} \tag{40}$$

$$x = r/a \tag{41}$$

$$c = b/a \tag{42}$$

Introducing these new quantities in (29) - (33) we obtain

$$\frac{\partial}{\partial \theta}\left[\frac{g(\tau)}{x^2} - x \frac{\partial \sigma}{\partial x}\right]^{m+1} = \left(x \frac{\partial \sigma}{\partial x}\right)^n \tag{43}$$

$$g(0) = \frac{2c^2}{c^2 - 1} \tag{44}$$

$$\sigma(x,0) = \frac{1 - \left(\frac{c}{x}\right)^2}{c^2 - 1} \tag{45}$$

$$\sigma(1,\tau) = -1 \tag{46}$$

$$\sigma(c,\tau) = 0 . \tag{47}$$

We now introduce two new unknown functions $w(x,\tau)$ and $y(x,\tau)$ by

$$w(x,\tau) = \frac{g(\tau)}{x^2} - x \frac{\partial \sigma(x,\tau)}{\partial x} \tag{48}$$

and

$$w(x,\tau) = \frac{2}{\lambda} \frac{c^{2/\lambda}}{c^{2/\lambda} - 1} . \left[\frac{\tau}{x^2} + \frac{y(x,\tau)}{x^2} - x^{-2/\lambda}\right] . \tag{49}$$

Originally we obtained the first term of (49) from the discussion in [3] on the stationary state and the other two terms from expanding $w(x,\tau) = y_{-1}(x)\tau + y_0(x) + y_1(x)\frac{1}{\tau} + \dots$. We now use (49) as an attempt, where it remains to prove that $y(x,\tau)$ is bounded. In fact, we will prove that $y(x,\tau)$ tends to a constant when τ tends to infinity.

Dividing (48) by x and integrating from $x = 1$ to $x = c$ we obtain, together with the boundary values for $\sigma(x,\tau)$,

$$g(\tau) = \frac{2c^2}{c^2-1} \cdot \left(1 + \int_1^c \frac{w(x,\tau)}{x}\,dx \right) . \tag{50}$$

After some elementary calculations we get a pure initial value problem for $y(x,\tau)$

$$\frac{\partial y(x,\tau)}{\partial \tau} = -1 + \frac{\left[1+x^{-2+2/\lambda}\left\{\frac{2c^2}{c^2-1}\int_1^c \frac{y(\xi,\tau)}{\xi^3}\,d\xi - y(x,\tau)\right\}\right]^n}{\left[1+\frac{1}{\tau}\left(y(x,\tau) - x^{2-2/\lambda}\right)\right]^m} \tag{51}$$

for $\tau > 0$,

$$y(x,0) = x^{2-2/\lambda}, \tag{52}$$

$$\frac{\partial y(x,0)}{\partial \tau} = \left(\frac{\dfrac{2c^2}{c^2-1}}{\dfrac{\frac{2}{\lambda}c^{2/\lambda}}{c^{2/\lambda}-1}} \right)^{\lambda} \cdot x^{2-2\lambda} - 1. \tag{53}$$

The desired functions $g(\tau)$ and $\sigma(x,\tau)$ may be expressed in terms of $y(x,\tau)$. The displacement is given by

$$g(\tau) = \frac{2}{\lambda} \cdot \frac{c^{2/\lambda}}{c^{2/\lambda}-1} \left(\tau + \frac{2c^2}{c^2-1} \int_1^c \frac{y(\xi,\tau)}{\xi^3}\,d\xi \right) \tag{54}$$

and the stress by

$$\sigma(x,\tau) = \frac{1-\left(\frac{c}{x}\right)^{2/\lambda}}{c^{2/\lambda}-1} + \frac{2}{\lambda}\,\frac{c^{2/\lambda}}{c^{2/\lambda}-1}\left\{ \frac{1-\frac{1}{x^2}}{1-\frac{1}{c^2}} \int_1^c \frac{y(\xi,\tau)}{\xi^3}\,d\xi - \int_1^x \frac{y(\xi,\tau)}{\xi^3}\,d\xi \right\} . \tag{55}$$

We assume $\lambda > 1$ (if $\lambda = 1$ we get the trivial case $y(x,t) \equiv 1$).

We first discuss the case of secondary creep ($m = 0$).

The denominator of (51) is then identically equal to 1 and the right hand member is a function $F[x, y, z] = -1 + [1 + x^{-2+2/\lambda} \cdot (z-y)]^n$ with

$$z = \frac{2c^2}{c^2 - 1} \int_1^c \frac{y(\xi)}{\xi^3} d\xi \ .$$

Since $\frac{2c^2}{c^2 - 1} \int_1^c \frac{d\xi}{\xi^3} = 1$ we find that z is a mean-value of $y(x)$ and that $F[x, y(x), z]$ is negative at the maximum points of $y(x)$ and positive at the minimum points. Using this property, it is possible to prove that the solution $y(x, \tau)$ of the initial-value problem tends to a constant y_∞ when τ tends to infinity.

We now discuss the more interesting case $m > 0$, where we will prove not only that $y(x, \tau) \rightarrow y_\infty$ when $\tau \rightarrow \infty$, but also obtain the value of the constant y_∞ in closed form.

By considering the sign of the derivative and the initial conditions we find $1 \leqslant y(x, \tau) \leqslant c^{2-2/\lambda}$ for all x and τ. In order to find the value of y_∞ we expand $y(x, \tau)$ in terms of $\frac{1}{\tau}$. We introduce a function $y^*(x, \tau)$ by

$$y(x, \tau) = y_\infty + \frac{1}{1+\tau} \frac{m}{n} \left\{ x^{4-4/\lambda} - y_\infty \ x^{2-2/\lambda} \right\} + y^*(x, \tau) \tag{56}$$

where

$$y_\infty = \frac{1}{\lambda - 2} \ \frac{c^{2-2/\lambda} - c^{2/\lambda}}{c^{2/\lambda} - 1} \qquad \text{if } \lambda \neq 2 \tag{57 a}$$

and

$$y_\infty = \frac{c}{c-1} \ln c \qquad \text{if } \lambda = 2. \tag{57 b}$$

Since

$$\int_1^c \frac{1}{x^3} \left\{ x^{4-4/\lambda} - y_\infty \ x^{2-2/\lambda} \right\} dx = 0 \tag{58}$$

we obtain from equation (51), after expanding in powers of $\frac{1}{\tau}$ and using the fact that $y^*(x, \tau)$ is bounded,

$$\frac{\partial y^*(x, \tau)}{\partial \tau} = -1 + \left[1 - \frac{m}{\tau} y^*(x, \tau) \right] \cdot \left[1 + \frac{x^{-2+2/\lambda} \left\{ z^*(\tau) - y^*(x, \tau) \right\}}{1 + \frac{1}{1+\tau} \frac{m}{n} (y_\infty - x^{2-2/\lambda})} \right]^n + 0 \left(\frac{1}{\tau^2} \right), \tag{59}$$

where

$$z^*(\tau) = \frac{2c^2}{c^2 - 1} \int_1^c \frac{y^*(x, \tau)}{x^3} dx. \tag{60}$$

We now consider only maxima with respect to x and get

$$\frac{\partial y^*(x,\tau)}{\partial \tau} < -\frac{m}{\tau} y^*(x,\tau) + \frac{C_0}{\tau^2} \tag{61}$$

for $x \in X_{Max}$ $(y^*(x,\tau))$, (the maximum points with respect to x) where C_0 is a fixed positive number. We now compare with the solution of the ordinary differential equation

$$\frac{dY}{d\tau} = -\frac{m}{\tau} Y(\tau) + \frac{C_0}{\tau^2}, \tag{62}$$

which is

$$Y(\tau) = \begin{cases} \dfrac{C_1}{\tau^m} + \dfrac{C_0}{m-1}\dfrac{1}{\tau} & m \neq 1 \\ \dfrac{C_1}{\tau} + \dfrac{C_0}{\tau}\ln \tau & m = 1 . \end{cases} \tag{63}$$

It may now be proved that

$$|y^*(x,\tau)| \leqslant \begin{cases} C/\tau^m & m < 1 \\ \dfrac{C}{\tau} \ln \tau & m = 1 \\ \dfrac{C}{\tau} & m > 1 . \end{cases} \tag{64}$$

We have thus proved that $y^*(x,\tau)$ tends to 0 when $\tau \to \infty$, which gives by equation (56) that $y(x,\tau) \to y_\infty$, with the value of y_∞ given by (57).

By a refined technique we obtain

$$y^*(x,\tau) = \begin{cases} 0(\dfrac{1}{\tau^m}) & 0 < m < 1 \\ y_1^\infty \cdot \dfrac{\ln(1+\tau)}{1+\tau} + 0(\dfrac{1}{\tau}) & m = 1 \\ \dfrac{y_1^\infty}{1+\tau} + 0(\dfrac{1}{\tau^m}) & 1 < m < 2 \\ \dfrac{y_1^\infty}{1+\tau} + 0(\dfrac{\ln \tau}{\tau^2}) & m = 2 \\ \dfrac{y_1^\infty}{1+\tau} + 0(\dfrac{1}{\tau^2}) & m > 2 \end{cases} \tag{65}$$

where y_1^∞ is a constant (for $m = 1$ it is easy to prove $y_1^\infty < 0$).

From equation (54) we now get the asymptotic displacement

$$g_{\infty}(\tau) = \tau \cdot \frac{2}{\lambda} \frac{c^{2/\lambda}}{c^{2/\lambda} - 1} + \frac{2}{\lambda(\lambda - 2)} \frac{c^{2} - c^{4/\lambda}}{(c^{2/\lambda} - 1)^{2}} \quad \text{if } \lambda \neq 2 \tag{66 a}$$

and

$$g_{\infty}(\tau) = \tau \cdot \frac{c}{c-1} + \left(\frac{c}{c-1}\right)^{2} \cdot \ln c \quad \text{if } \lambda = 2. \tag{66 b}$$

The value of $g_{\infty}(0)$ is of special interest and is given below expanded in terms of $\Delta = c - 1$.

$$g_{\infty}(0) = \frac{1}{\Delta} \left[1 + \frac{3}{2} \Delta + \frac{1}{4} \Delta^{2} + \frac{(\lambda-1)^{2}}{3\lambda^{2}} \Delta^{2} + 0(\Delta^{3}) \right]. \tag{67}$$

This result will be compared with the initial value (44)

$$g(0) = \frac{2c^{2}}{c^{2} - 1} = \frac{1}{\Delta} \left[1 + \frac{3}{2} \Delta + \frac{1}{4} \Delta^{2} + 0(\Delta^{3}) \right]. \tag{68}$$

From (55) and (56) we get the asymptotic stress

$$\sigma_{\infty}(x) = \frac{1 - \left(\frac{c}{x}\right)^{2/\lambda}}{c^{2/\lambda} - 1}, \tag{69}$$

which is a well-known result.

For $m > 2$ we obtain

$$g(\tau) = \frac{2}{\lambda} \frac{c^{2/\lambda}}{c^{2/\lambda} - 1} \cdot \left[\tau + y_{\infty} + \frac{y_{1}^{\infty}}{1+\tau} + 0\left(\frac{1}{\tau^{2}}\right) \right] \tag{70}$$

and

$$\sigma(x, \tau) = \frac{1 - \left(\frac{c}{x}\right)^{2/\lambda}}{c^{2/\lambda} - 1} + \frac{1}{1+\tau} \frac{m}{n} \frac{c^{2/\lambda}}{c^{2/\lambda} - 1} \cdot \left[y_{\infty} (1 - x^{-2/\lambda}) - \frac{x^{2-4/\lambda} - 1}{\lambda - 2} \right] + 0\left(\frac{1}{\tau^{2}}\right)$$

if $\lambda \neq 2$, (71 a)

$$\sigma(x, \tau) = \frac{1 - \frac{c}{x}}{c - 1} + \frac{1}{1+\tau} \cdot \frac{m}{2(m+1)} \cdot \frac{c}{c-1} \cdot$$

$$\cdot \left[\frac{c}{c-1} \cdot \ln c \cdot \left(1 - \frac{1}{x}\right) - \ln x \right] + 0\left(\frac{1}{\tau^{2}}\right) \tag{71 b}$$

if $\lambda = 2$

Corresponding formulas may be derived for other values of m.

4. Numerical procedure.

The following initial-value problem is to be solved in the semi-infinite strip $0 \leqslant x \leqslant 1, \quad t \geqslant 0$:

$$\frac{\partial y(x,t)}{\partial t} = F\left[x, t, y(x,t), \int_0^1 \mu(\xi) y(\xi, t) d\xi\right] \tag{72}$$

$$y(x,0) = y_0(x) \tag{73}$$

The functions $F[x,t,y,z]$ and y (x) have continuous (partial) derivatives of the first order and $\mu(x)$ is non-negative and continuous. By considering the Banach space of continuous functions $y(x)$ on $0 \leqslant x \leqslant 1$ with the maximum norm we can use the existence and uniqueness theorem for the ordinary differential equation on a Banach space (see Kato [5]) to prove the existence of a unique solution of the integro-differential equation in the region $0 \leqslant x \leqslant 1, \; 0 \leqslant t \leqslant T$.

In order to show that $y(x,t) \to y_\infty$ when $t \to \infty$ we introduce the property C by the following

Definition. The function $F[x,t,y,z]$ has the property C if:

1^0. $F[x,t,y,z]$ is independent of t,

and

2^0. with $z = \int_0^1 \mu(x) y(x) dx$ the following inequalities hold for all $y(x)$

$$\begin{array}{lll} F[x,y,z] < 0 & \text{for} & x \in X_{Max}(y) \\ F[x,y,z] > 0 & \text{for} & x \in X_{Min}(y) \\ F[x,y,z] = 0 & \text{if} & y(x) \text{ is constant} \end{array}$$

where

$$X_{Max}(y) = \left\{ x \;\middle|\; y(x) = \underset{0 \leqslant x \leqslant 1}{\mathrm{Max}}\, y(x), \; y(x) \text{ not constant} \right\}$$

$$X_{Min}(y) = \left\{ x \;\middle|\; y(x) = \underset{0 \leqslant x \leqslant 1}{\mathrm{Min}}\, y(x), \; y(x) \text{ not constant} \right\}.$$

We then have the following theorem, which is proved in an unpublished report.

Theorem. If F has the property C and if $y(x,t)$ is the solution of the initial value problem then $\underset{0 \leqslant x \leqslant 1}{\mathrm{Max}}\, y(x,t)$ is a decreasing function of t. If in addition $F_y'[x, y(x,t), z(t)] \leqslant -\epsilon < 0$ for all $x \in [0,1]$ and all $t \geqslant 0$ then $y(x,t)$ tends to a constant when $t \to \infty$.

The numerical procedure used to obtain the solution is the Runge-Kutta method. To determine the integral we need a quadrature formula, that preserves the property $\mathfrak{C}$ to the discrete case. We therefore construct a formula analogous to Simpson's rule by requiring

$$\int_{a-h}^{a+h} \frac{y(x)}{x^3} dx = b_{-1}\, y(a-h) + b_0 y(a) + b_1 y(a+h) + R(y), \tag{74}$$

with $R(y) = 0$ for $y = 1$, x and x^2. We get

$$b_{-1} = \frac{h(a+h)}{(a^2-h^2)^2} - \frac{a}{h}\cdot\frac{1}{a^2-h^2} + \frac{1}{2h^2}\ln\frac{a+h}{a-h},$$

$$b_0 = \frac{2a}{h}\;\frac{1}{a^2-h^2} - \frac{1}{h^2}\ln\frac{a+h}{a-h},\quad \text{and} \tag{75}$$

$$b_1 = \frac{h(a-h)}{(a^2-h^2)^2} - \frac{a}{h}\,\frac{1}{a^2-h^2} + \frac{1}{2h^2}\ln\frac{a+h}{a-h}\,.$$

The quadrature error for a fixed interval is $0(h^4)$ while the truncation error in one step of length k is $0(k^5)$ with the Runge-Kutta formula (see e.g. Henrici [6]). The total error is thus $O(k^4) + O(h^4)$. Because of the property C the error growth is at most linear in t.

It was found satisfactory to use the time step 0.01 and 10 sub-intervals for x.

5. *Discussion of results.*

A quantity of main interest is the displacement u(a, t) of the inner surface of the vessel, see Fig. 1.

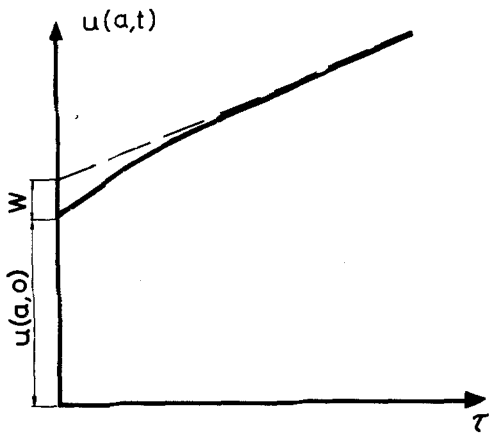

Fig. 1. The displacement u(a, t) of the inner surface as function of the transformed time τ.

The displacement is obtained from equations (36), (38) and (54),

$$u(a,t) = R_\infty\,.\,\left[\tau + z(\tau)\right] \tag{76}$$

where

$$R_\infty = \frac{3}{4}\,\frac{pa}{E}\,\frac{2}{\lambda}\;\frac{c^{2/\lambda}}{c^{2/\lambda}-1} \tag{77}$$

and

$$z(\tau) = \frac{2c^2}{c^2-1}\int_1^c \frac{y(\xi,\tau)}{\xi^3}\,d\xi\,. \tag{78}$$

The additional displacement W, due to statical primary creep, is obtained from equations (56) and (57) if $m > 0$. (If $m = 0$ we have to use a numerical computation to obtain y_∞.) For $m > 0$ we have $W = R_\infty \cdot [y_\infty - z(0)]$ or

$$W = \frac{3}{4}\frac{pa}{E}\left\{\frac{2}{\lambda(\lambda-2)}\frac{c^2 - c^{4/\lambda}}{(c^{2/\lambda}-1)^2} - \frac{2c^2}{c^2-1}\right\} \qquad \text{if } \lambda \neq 2 \tag{79 a}$$

and

$$W = \frac{3}{4}\frac{pa}{E}\left\{\left(\frac{c}{c-1}\right)^2 \cdot \ln c - \frac{2c^2}{c^2-1}\right\} \qquad \text{if } \lambda = 2, \tag{79 b}$$

or expanded in terms of $\Delta = c - 1$

$$W = \frac{3}{4} \cdot \frac{pa}{E} \cdot \frac{(\lambda-1)^2}{3 \cdot \lambda^2}\Delta + 0(\Delta^2). \tag{80}$$

This result is not in complete agreement with the result of Rosengren [3]

$$W_{m=1} = \frac{3}{4}\frac{pa}{E}\,\frac{(\lambda-1)^2(\lambda+1)}{3 \cdot \lambda^2}\,\Delta + 0(\Delta^2) , \tag{81}$$

obtained after solution by the perturbation method for thinwalled vessels.

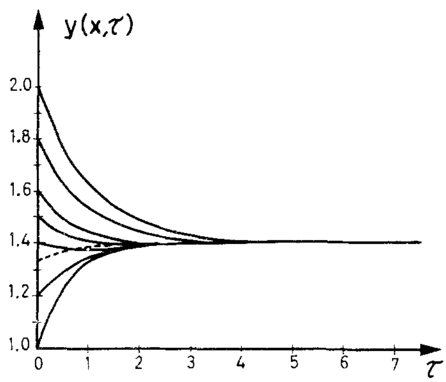

Fig. 2. Secondary creep.
The function $y(x,\tau)$ is given for $x = 1.0, 1.2, 1.4, 1.5, 1.6, 1.8$ and 2.0 and is represented by solid curves, while $z(\tau)$ is represented by a dashed curve. The parameters are $m=0$, $n=2$ and $c = 2$, giving $y(x,0) = x$ and $z(0) = 4/3$. The computed value of y_∞ is 1.3997.

We are especially interested in how $z(\tau)$ approaches the constant y_∞. For $m > 0$ we obtain from (56) and (65) that this convergence is of the type $1/\tau$, while the computations show an exponential convergence if $m = 0$. In figures 2 and 3 we give examples of this.

A very simple measure of the duration of the transient period is given by the quantity τ^* introduced by Hult [2] and defined from the rate of growth of the inner radius (Fig. 4.).

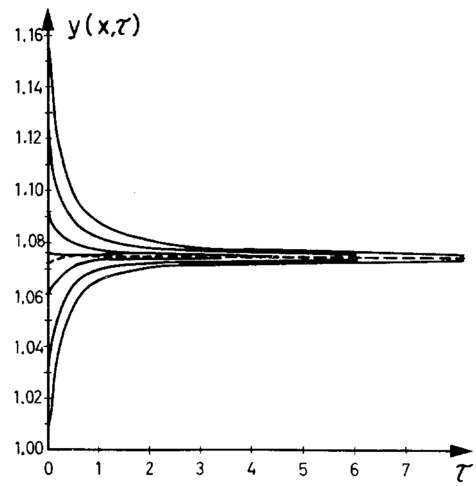

Fig.3. Primary creep.
The function $y(x,\tau)$ is given for $x = 1.00$, 1.02, 1.04, 1.05, 1.06, 1.08 and 1.10 and is represented by solid curves, while $z(\tau)$ is represented by a dashed curve. The parameters are $m = 1$, $n = 8$ and $c = 1.1$, giving $y(x,0) = x^{3/2}$, $z(0) = 1.072577$ and $y_\infty = 1.074404$.

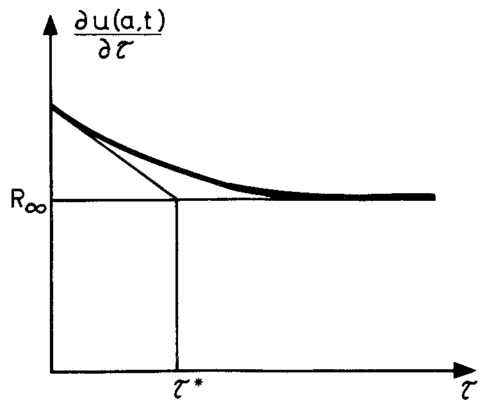

Fig.4. The rate of growth of the inner radius with respect to the transformed time τ as function of τ.

Differentiating (76) we obtain

$$\frac{\partial u(a,t)}{\partial \tau} = R_\infty \cdot \left[1 + z'(\tau)\right], \tag{82}$$

and τ^* is obtained from

$$\tau^* = - z'(0)/z''(0). \tag{83}$$

From (51)-(53) we get the following expression for the duration of the transient period, expressed in seconds,

$$t^* = \left(\frac{1}{2E}\right)^{m+1} \frac{1}{K}\left(\frac{1}{p\sqrt{3}}\right)^{n-(m+1)} \cdot \frac{(m+2)^{m+1}}{(m+1)^{m+2}} .$$

$$\left\{ \frac{(1-c^{-2\lambda}).(1-c^{-2})^{\lambda} - \lambda^{1-\lambda}(1-c^{-2/\lambda})^{-\lambda} \; . \; (1-c^{-2})^{2\lambda+1}}{\frac{\lambda^2}{2\lambda-1}(1-c^{-2})(1-c^{2-4\lambda}) - (1-c^{-2\lambda})^2} \right\}^{m+1} \qquad (84)$$

It is essential that this t^* has an immediate connection to half-life periods only for $m = 0$, since it is only in this case the function is exponential.

APPENDIX

In this appendix the results for the spherical case are given. Definitions:

$$G(\tau) = \frac{2}{1-\mu} \frac{E}{p} \frac{1}{a^3} F(t) \qquad (85)$$

$$\theta = (m+1) \frac{2^{-n}}{(1-\mu)^{m+1}} . K . p^n . \left(\frac{E}{p}\right)^{m+1} . t \qquad (86)$$

$$\tau = \theta^{\frac{1}{m+1}} . \left(\frac{3}{\lambda} \frac{c^{3/\lambda}}{c^{3/\lambda} - 1}\right)^{\lambda-1} \qquad (87)$$

Differential equation

$$\frac{\partial}{\partial\theta} \left[\frac{G(\tau)}{x^3} - x \frac{\partial\sigma}{\partial x} \right]^{m+1} = \left(x \frac{\partial\sigma}{\partial x} \right)^n \qquad (88)$$

Initial conditions

$$G(0) = \frac{3c^3}{c^3 - 1} \qquad (89)$$

$$\sigma(x, 0) = \frac{1 - \left(\frac{c}{x}\right)^3}{c^3 - 1} . \qquad (90)$$

The boundary conditions (46) and (47) remain unchanged.

$$w(x, \tau) = \frac{G(\tau)}{x^3} - x \frac{\partial\sigma(x, \tau)}{\partial x} \qquad (91)$$

$$w(x, \tau) = \frac{3}{\lambda} \frac{c^{3/\lambda}}{c^{3/\lambda} - 1} . \left[\frac{\tau}{x^3} + \frac{y(x, \tau)}{x^3} - x^{-3/\lambda} \right] \qquad (92)$$

$$G(\tau) = \frac{3c^3}{c^3 - 1} \left(1 + \int_1^c \frac{w(x, \tau)}{x} dx\right) \qquad (93)$$

The new initial value problem is defined by

$$\frac{\partial y(x,\tau)}{\partial \tau} = -1 + \frac{\left[1 + x^{-3+3/\lambda}\left\{\dfrac{3c^3}{c^3-1}\displaystyle\int_1^c \frac{y(\xi,\tau)}{\xi^4}\,d\xi - y(x,\tau)\right\}\right]^n}{\left[1 + \dfrac{1}{\tau}\left\{y(x,\tau) - x^{3-3/\lambda}\right\}\right]^m} \tag{94}$$

for $\tau > 0$,

$$y(x,0) = x^{3-3/\lambda}, \qquad \text{and} \tag{95}$$

$$\frac{\partial y(x,0)}{\partial \tau} = \left(\frac{\dfrac{3c^3}{c^3-1}}{\dfrac{\frac{3}{\lambda}c^{3/\lambda}}{c^{3/\lambda}-1}}\right)^{\lambda} \cdot x^{3-3\lambda} - 1. \tag{96}$$

The desired functions $G(\tau)$ and $\sigma(x,\tau)$ are easily expressed in terms of $y(x,\tau)$.

$$G(\tau) = \frac{3}{\lambda}\,\frac{c^{3/\lambda}}{c^{3/\lambda}-1}\left(\tau + \frac{3c^3}{c^3-1}\int_1^c \frac{y(\xi,\tau)}{\xi^4}\,d\xi\right) \tag{97}$$

$$\sigma(x,\tau) = \frac{1-\left(\frac{c}{x}\right)^{3/\lambda}}{c^{3/\lambda}-1} + \frac{3}{\lambda}\,\frac{c^{3/\lambda}}{c^{3/\lambda}-1}\,\cdot$$

$$\cdot\left\{\frac{1-\dfrac{1}{x^3}}{1-\dfrac{1}{c^3}}\int_1^c \frac{y(\xi,\tau)}{\xi^4}\,d\xi - \int_1^x \frac{y(\xi,\tau)}{\xi^4}\,d\xi\right\}. \tag{98}$$

For $m > 0$ we obtain

$$y(x,\tau) = y_\infty + \frac{1}{1+\tau}\,\frac{m}{n}\left\{x^{6-6/\lambda} - y_\infty\, x^{3-3/\lambda}\right\} + y^*(x,\tau), \tag{99}$$

where

$$y_\infty = \frac{1}{\lambda-2}\,\frac{c^{3-3/\lambda} - c^{3/\lambda}}{c^{3/\lambda}-1} \qquad \text{if } \lambda \neq 2, \tag{100 a}$$

and

$$y_\infty = \frac{3}{2}\,\frac{c^{3/2}}{c^{3/2}-1}\,\ln c \qquad \text{if } \lambda = 2, \tag{100 b}$$

with $y^*(x, \tau)$ bounded as in the cylindrical case (eq. 64).

The additional displacement is given by

$$W = \frac{1-\mu}{2}\frac{pa}{E}\left\{\frac{3}{\lambda(\lambda-2)}\cdot\frac{c^3 - c^{6/\lambda}}{(c^{3/\lambda} - 1)^2} - \frac{3c^3}{c^3 - 1}\right\} \text{ if } \lambda \neq 2 \qquad (101\ a)$$

and

$$W = \frac{1-\mu}{2}\frac{pa}{E}\left\{\frac{9}{4}\frac{c^3}{(c^{3/2} - 1)^2}\cdot \ln c - \frac{3c^3}{c^3 - 1}\right\} \text{ if } \lambda = 2, \qquad (101\ b)$$

or expanded in terms of $\Delta = c-1$

$$W = \frac{1-\mu}{2}\cdot\frac{pa}{E}\cdot\frac{3}{4}\frac{(\lambda-1)^2}{\lambda^2}\Delta + 0(\Delta^2). \qquad (102)$$

The time t^* is given by Hult [2] in the spherical case.

For those who wish to solve equation (94) we give the corresponding quadrature coefficients for the evaluation of

$$\int_{a-h}^{a+h} \frac{y(x)}{x^4}\, dx.$$

$$\begin{cases} b_{-1} = \frac{1}{3}h\,\frac{a^2 + 4ah + 3h^2}{(a^2 - h^2)^3} \\ b_0 = \frac{4}{3}h\,\frac{1}{(a^2 - h^2)^2} \\ b_1 = \frac{1}{3}h\,\frac{a^2 - 4ah + 3h^2}{(a^2 - h^2)^3} \end{cases} \qquad (103)$$

Acknowledgements

I wish to thank Professor Heinz-Otto Kreiss for his invaluable advice and criticism. I also thank Professor Jan Hult and Dr. Göran Rosengren for stimulating discussions and their kind interest in my work.

Bibliography

1. J.F.Besseling, "Investigation of Transient Creep in Thick-walled Tubes under Axially Symmetric Loading", IUTAM Colloquium on Creep in Structures, Stanford 1960, Proceedings, Springer, Berlin 1962, 174-194.

2. J.Hult, "Primary Creep in Thickwalled Spherical Shells", Trans. Chalmers Univ. of Technol. No. 264, Gothenburg 1963, 26 pp.

3. G.Rosengren, "Primary Creep in Cylindrical Pressure Vessels", Archiwum Mechaniki Stosowanej, 4,16 (1964), 959-972.

4. F.K.G.Odqvist and J.Hult, "Kriechfestigkeit metallischer Werkstoffe", Springer, Berlin 1962, 303 pp.

5. T. Kato, "Nonlinear Evolution Equations in Banach Spaces", Proceedings of Symposia in Applied Mathematics, Volume XVII, American Mathematical Society, Providence, Rhode Island, 1965, 50-67.

6. P. Henrici, "Discrete Variable Methods in Ordinary Differential Equations", Wiley, New York 1962, 407 pp.

[Received December 7, 1967]